

# PROOF OF A CONJECTURE ON 6-COLORED GENERALIZED FROBENIUS PARTITIONS

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ABSTRACT. Let  $c\phi_k(n)$  be the  $k$ -colored generalized Frobenius partition function. By employing the generating function of  $c\phi_6(3n+1)$  found by Hirschhorn, we prove that  $c\phi_6(27n+16) \equiv 0 \pmod{243}$ . This confirms a conjecture of E.X.W. Xia. We also find a congruence relation  $c\phi_6(81n+61) \equiv 3c\phi_6(9n+7) \pmod{243}$ . Moreover, we show that  $c\phi_6(81n+61) \equiv 0 \pmod{81}$ ,  $c\phi_6(243n+142) \equiv 0 \pmod{243}$  and  $c\phi_6(729n+547) \equiv 0 \pmod{243}$ . We further conjecture that for  $n \geq 0$ ,  $c\phi_6(243n+142) \equiv 0 \pmod{729}$ .

## 1. INTRODUCTION

In his 1984 AMS Memoir, Andrews [1] introduced the concept of generalized Frobenius partitions. For any positive integer  $k$ , let  $c\phi_k(n)$  denote the number of  $k$ -colored generalized Frobenius partition function of  $n$ . The generating function of  $c\phi_k(n)$  is given by

$$\sum_{n=0}^{\infty} c\phi_k(n)q^n = \frac{1}{(q; q)_{\infty}^k} \sum_{m_1, \dots, m_{k-1}=-\infty}^{\infty} q^{Q(m_1, \dots, m_{k-1})},$$

where

$$Q(m_1, \dots, m_{k-1}) = \sum_{i=1}^{k-1} m_i^2 + \sum_{1 \leq i < j \leq k-1} m_i m_j.$$

In particular, if  $k = 6$ , Baruah and Sarmah [2] proved that

$$\begin{aligned} \sum_{n=0}^{\infty} c\phi_6(n)q^n &= \frac{1}{(q; q)_{\infty}^6} \left( \varphi^3(q)\varphi(q^2)\varphi(q^6) + 24q\psi^3(q)\psi(q^2)\psi(q^3) \right. \\ &\quad \left. + 4q^2\varphi^3(q)\psi(q^4)\psi(q^{12}) \right), \end{aligned} \tag{1.1}$$

where as usual,  $\varphi(q)$  and  $\psi(q)$  are Ramanujan's theta functions, namely (see [3], for example)

$$\varphi(q) = \sum_{n=0}^{\infty} q^{n^2}, \quad \psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}.$$

They also established the 2- and 3- dissections of (1.1), from which they proved some interesting Ramanujan-type congruences: for  $n \geq 0$ ,

$$\begin{aligned} c\phi_6(2n+1) &\equiv 0 \pmod{4}, \\ c\phi_6(3n+1) &\equiv 0 \pmod{9}, \end{aligned} \tag{1.2}$$

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and

$$c\phi_6(3n+2) \equiv 0 \pmod{9}. \quad (1.3)$$

They also proposed the following conjecture: for  $n \geq 0$ ,

$$c\phi_6(3n+2) \equiv 0 \pmod{27}. \quad (1.4)$$

By utilizing the generating function for  $c\phi_6(3n+2)$  given by Baruah and Sarmah [2] and the  $(p, k)$ -parametrization of theta functions due to Alaca and Williams, Xia [8] proved (1.4) and he further conjectured that

$$c\phi_6(9n+7) \equiv 0 \pmod{27}, \quad (1.5)$$

and

$$c\phi_6(27n+16) \equiv 0 \pmod{243}. \quad (1.6)$$

By using some known  $q$  series identities, Hirschhorn [6] gave a new presentation of 3-dissections of (1.1), from which both (1.2) and (1.3) follows readily. He also proved (1.5) but left (1.6) still open.

In this note, by employing the generating function for  $c\phi_6(3n+1)$  found by Hirschhorn [6], we obtain the following result.

**Theorem 1.** *For any integer  $n \geq 0$ , we have*

$$c\phi_6(27n+16) \equiv 0 \pmod{243}, \quad (1.7)$$

$$c\phi_6(81n+61) \equiv 0 \pmod{81}, \quad (1.8)$$

$$c\phi_6(243n+142) \equiv 0 \pmod{243}, \quad (1.9)$$

$$c\phi_6(729n+547) \equiv 0 \pmod{243} \quad (1.10)$$

and the congruence relation

$$c\phi_6(81n+61) \equiv 3c\phi_6(9n+7) \pmod{243}.$$

Since

$$c\phi_6(16) = 2 \times 3^5 \times 1222049, \quad c\phi_6(61) = 2^2 \times 3^4 \times 19 \times 701612098458871$$

and

$$\begin{aligned} c\phi_6(547) &= 2^5 \times 3^5 \times 409 \times 6661 \times 3949235117518927056389 \\ &\quad \times 20029030597437898896898971631, \end{aligned}$$

we see that (1.7), (1.8) and (1.10) are best possible in the sense that the modulus cannot be replaced by higher powers of 3. However, based on some numerical evidences, we believe that (1.9) can be improved. We propose the following conjecture for research in the future.

**Conjecture 1.** *For any integer  $n \geq 0$ , we have*

$$c\phi_6(243n+142) \equiv 0 \pmod{729}.$$

## 2. PRELIMINARIES

In this section, we present some 3-dissection identities, which will play a key role in our proof.

It is easy to see that

$$\begin{aligned}\varphi(q) &= \frac{(q^2; q^2)_\infty^5}{(q; q)_\infty^2 (q^4; q^4)_\infty^2}, & \varphi(-q) &= \frac{(q; q)_\infty^2}{(q^2; q^2)_\infty}, \\ \psi(q) &= \frac{(q^2; q^2)_\infty^2}{(q; q)_\infty}, & \psi(-q) &= \frac{(q; q)_\infty (q^4; q^4)_\infty}{(q^2; q^2)_\infty}.\end{aligned}$$

Moreover, let

$$\begin{aligned}X(q) &:= \sum_{n=-\infty}^{\infty} q^{3n^2+2n} = \frac{(q^2; q^2)_\infty^2 (q^3; q^3)_\infty (q^{12}; q^{12})_\infty}{(q; q)_\infty (q^4; q^4)_\infty (q^6; q^6)_\infty}, \\ Y(q) &:= \sum_{n=-\infty}^{\infty} q^{n(3n+1)/2} = \frac{(q^2; q^2)_\infty (q^3; q^3)_\infty^2}{(q; q)_\infty (q^6; q^6)_\infty}.\end{aligned}$$

We have the following 3-dissection identities (see [4, Corollay, p.49]):

$$\varphi(q) = \varphi(q^9) + 2qX(q^3), \quad \psi(q) = Y(q^3) + q\psi(q^9). \quad (2.1)$$

Let

$$a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} = 1 + 6 \sum_{n \geq 0} \left( \frac{q^{3n+1}}{1 - q^{3n+1}} - \frac{q^{3n+2}}{1 - q^{3n+2}} \right).$$

As shown in [5], we have

$$a(q) = a(q^3) + 6q \frac{(q^9; q^9)_\infty^3}{(q^3; q^3)_\infty}.$$

This implies

$$a(q)^5 \equiv a(q^3)^5 + 3qa(q^3)^4 \frac{(q^9; q^9)_\infty^3}{(q^3; q^3)_\infty} + 9q^2a(q^3)^3 \frac{(q^9; q^9)_\infty^6}{(q^3; q^3)_\infty^2} \pmod{27}, \quad (2.2)$$

and

$$a(q)^6 \equiv a(q^3)^6 + 9qa(q^3)^5 \frac{(q^9; q^9)_\infty^3}{(q^3; q^3)_\infty} \pmod{27}. \quad (2.3)$$

**Lemma 1.** *Let  $p$  be a prime and  $\alpha$  be a positive integer. Then*

$$(q; q)_\infty^{p^\alpha} \equiv (q^p; q^p)_\infty^{p^{\alpha-1}} \pmod{p^\alpha}.$$

*Proof.* Note that for any prime  $p$ , we have

$$\binom{p}{k} = \frac{p}{k} \cdot \binom{p-1}{k-1} \equiv 0 \pmod{p}, \quad 1 \leq k \leq p-1.$$

By the binomial theorem, we have

$$(1-x)^p = 1 - px + \cdots + p(-x)^{p-1} + (-x)^p \equiv 1 - x^p \pmod{p}.$$

Hence we have

$$(q; q)_\infty^p \equiv (q^p; q^p)_\infty \pmod{p}.$$

This proves the lemma for  $\alpha = 1$ . Suppose for some  $\alpha \geq 2$  we have

$$(q; q)_\infty^{p^{\alpha-1}} \equiv (q^p; q^p)_\infty^{p^{\alpha-2}} \pmod{p^{\alpha-1}},$$

then there exists a series  $f(q)$  with integer coefficients such that

$$(q; q)_\infty^{p^{\alpha-1}} = (q^p; q^p)_\infty^{p^{\alpha-2}} + p^{\alpha-1} f(q).$$

Again by the binomial theorem, we deduce that

$$(q; q)_\infty^{p^\alpha} = \left( (q^p; q^p)_\infty^{p^{\alpha-2}} + p^{\alpha-1} f(q) \right)^p \equiv (q^p; q^p)_\infty^{p^{\alpha-1}} \pmod{p^\alpha}.$$

By induction on  $\alpha$ , we complete our proof.  $\square$

The following 3-dissection identity is also useful in our arguments.

**Lemma 2.** *We have*

$$(q; q)_\infty^3 = (q^3; q^3)_\infty a(q^3) - 3q(q^9; q^9)_\infty^3. \quad (2.4)$$

*Proof.* By Jacobi's identity [3, Theorem 1.3.9], we have

$$(q; q)_\infty^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2}.$$

Note that  $\frac{n(n+1)}{2} \equiv 0 \pmod{3}$  if and only if  $n \equiv 0 \pmod{3}$  or  $n \equiv 2 \pmod{3}$ . And  $\frac{n(n+1)}{2} \equiv 1 \pmod{3}$  if and only if  $n \equiv 1 \pmod{3}$ . Hence we have the following 3-dissection identity

$$(q; q)_\infty^3 = P(q^3) + qR(q^3).$$

We have

$$\begin{aligned} P(q^3) &= \sum_{m=0}^{\infty} (-1)^{3m} (6m+1) q^{3m(3m+1)/2} + \sum_{m=0}^{\infty} (-1)^{3m+2} (6m+5) q^{(3m+2)(3m+3)/2} \\ &= \sum_{m=0}^{\infty} (-1)^m (6m+1) q^{3m(3m+1)/2} + \sum_{m=-\infty}^{-1} (-1)^m (6m+1) q^{3m(3m+1)/2} \\ &= \sum_{m=-\infty}^{\infty} (-1)^m (6m+1) q^{3m(3m+1)/2}. \end{aligned}$$

Replacing  $q^3$  by  $q$ , we obtain

$$P(q) = \sum_{m=-\infty}^{\infty} (-1)^m (6m+1) q^{m(3m+1)/2}.$$

From [7] we know that

$$P(q) = (q; q)_\infty \left( 1 + 6 \sum_{n \geq 0} \left( \frac{q^{3n+1}}{1 - q^{3n+1}} - \frac{q^{3n+2}}{1 - q^{3n+2}} \right) \right) = (q; q)_\infty a(q).$$

Again, we have

$$qR(q^3) = \sum_{m=0}^{\infty} (-1)^{3m+1} (6m+3) q^{(3m+1)(3m+2)/2}.$$

Dividing both sides by  $q$  and replacing  $q^3$  by  $q$ , we deduce that

$$R(q) = -3 \sum_{m=0}^{\infty} (-1)^m (2m+1) q^{3m(m+1)/2} = -3(q^3; q^3)_\infty^3.$$

$\square$

## 3. PROOF OF THEOREM 1

From [6] we find

$$\begin{aligned}
 & \sum_{n \geq 0} c\phi_6(3n+1)q^n \\
 &= 9 \left( \frac{(q^2; q^2)_\infty^5 (q^3; q^3)_\infty^6}{(q; q)_\infty^{22} (q^4; q^4)_\infty^2} \left( 2a(q)^5 \frac{(q^3; q^3)_\infty^3}{(q; q)_\infty} + 189qa(q)^2 \frac{(q^3; q^3)_\infty^{12}}{(q; q)_\infty^4} \right) \right. \\
 & \quad + \frac{(q^3; q^3)_\infty^9 (q^4; q^4)_\infty (q^6; q^6)_\infty^2}{(q; q)_\infty^{23} (q^2; q^2)_\infty (q^{12}; q^{12})_\infty} \\
 & \quad \times \left( 2a(q)^6 + 378qa(q)^3 \frac{(q^9; q^9)_\infty^9}{(q; q)_\infty^3} + 1458q^2 \frac{(q^3; q^3)_\infty^{18}}{(q; q)_\infty^6} \right) \\
 & \quad \left. - \frac{(q^3; q^3)_\infty^9 (q^{12}; q^{12})_\infty^2}{(q; q)_\infty^{23} (q^6; q^6)_\infty} \left( 36qa(q)^5 \frac{(q^3; q^3)_\infty^3}{(q; q)_\infty} + 1944q^2 a(q)^2 \frac{(q^3; q^3)_\infty^{12}}{(q; q)_\infty^4} \right) \right). \tag{3.1}
 \end{aligned}$$

By Lemma 1 we have

$$(q; q)_\infty^3 \equiv (q^3; q^3)_\infty \pmod{3}, \quad (q; q)_\infty^{27} \equiv (q^3; q^3)_\infty^9 \pmod{27}.$$

Hence

$$\begin{aligned}
 & \sum_{n=0}^{\infty} c\phi_6(3n+1)q^n \\
 & \equiv 18 \left( a(q)^5 \frac{(q^2; q^2)_\infty^5 (q^3; q^3)_\infty^9}{(q; q)_\infty^{23} (q^4; q^4)_\infty^2} + a(q)^6 \frac{(q^3; q^3)_\infty^9 (q^4; q^4)_\infty (q^6; q^6)_\infty^2}{(q; q)_\infty^{23} (q^2; q^2)_\infty (q^{12}; q^{12})_\infty} \right. \\
 & \quad \left. - 18qa(q)^5 \frac{(q^3; q^3)_\infty^{12} (q^{12}; q^{12})_\infty^2}{(q; q)_\infty^{24} (q^6; q^6)_\infty} \right) \\
 & \equiv 18 \left( a(q)^5 \frac{(q^2; q^2)_\infty^5 (q; q)_\infty^4}{(q^4; q^4)_\infty^2} + a(q)^6 \frac{(q; q)_\infty^4 (q^4; q^4)_\infty (q^6; q^6)_\infty^2}{(q^2; q^2)_\infty (q^{12}; q^{12})_\infty} \right. \\
 & \quad \left. - 18qa(q)^5 \frac{(q^3; q^3)_\infty^{12} (q^{12}; q^{12})_\infty^2}{(q^3; q^3)_\infty^8 (q^6; q^6)_\infty} \right) \pmod{243}. \tag{3.2}
 \end{aligned}$$

Note that

$$\frac{(q^2; q^2)_\infty^5 (q; q)_\infty^4}{(q^4; q^4)_\infty^2} = \frac{(q^2; q^2)_\infty^5}{(q; q)_\infty^2 (q^4; q^4)_\infty^2} \cdot (q; q)_\infty^6 = \varphi(q)(q; q)_\infty^6, \tag{3.3}$$

$$\frac{(q; q)_\infty^4 (q^4; q^4)_\infty}{(q^2; q^2)_\infty} = \frac{(q; q)_\infty (q^4; q^4)_\infty}{(q^2; q^2)_\infty} \cdot (q; q)_\infty^3 = \psi(-q)(q; q)_\infty^3. \tag{3.4}$$

By (2.1), (2.2), (3.3) and Lemma 2, we have

$$\begin{aligned}
 & a(q)^5 \frac{(q^2; q^2)_\infty^5 (q; q)_\infty^4}{(q^4; q^4)_\infty^2} \\
 & \equiv \left( a(q^3)^5 + 3qa(q^3)^4 \frac{(q^9; q^9)_\infty^3}{(q^3; q^3)_\infty} + 9q^2 a(q^3)^3 \frac{(q^9; q^9)_\infty^6}{(q^3; q^3)_\infty^2} \right) \left( \varphi(q^9) + 2qX(q^3) \right) \\
 & \quad \left( a(q^3)^2 (q^3; q^3)_\infty^2 - 6qa(q^3) (q^3; q^3)_\infty (q^9; q^9)_\infty^3 + 9q^2 (q^9; q^9)_\infty^6 \right) \pmod{27},
 \end{aligned}$$

Extracting the terms of the form  $q^{3n+2}$  in both sides, we obtain after simplification that

$$q^2 I(q^3) \equiv -6q^2 a(q^3)^6 X(q^3)(q^3; q^3)_\infty (q^9; q^9)_\infty^3 \pmod{27}. \quad (3.5)$$

Thus

$$I(q) \equiv -6a(q)^6 X(q)(q; q)_\infty (q^3; q^3)_\infty^3 \pmod{27}.$$

Similarly, by (2.1), (2.3), (3.4) and Lemma 2 we have

$$\begin{aligned} & a(q)^6 \frac{(q; q)_\infty^4 (q^4; q^4)_\infty (q^6; q^6)_\infty^2}{(q^2; q^2)_\infty (q^{12}; q^{12})_\infty} \\ & \equiv \frac{(q^6; q^6)_\infty^2}{(q^{12}; q^{12})_\infty} \left( a(q^3)^6 + 9qa(q^3)^5 \frac{(q^9; q^9)_\infty^3}{(q^3; q^3)_\infty} \right) \\ & \quad \left( Y(-q^3) - q\psi(-q^9) \right) \left( a(q^3)(q^3; q^3)_\infty - 3q(q^9; q^9)_\infty^3 \right) \pmod{27}. \end{aligned}$$

Extracting the terms of the form  $q^{3n+2}$  in both sides, we obtain

$$q^2 J(q^3) \equiv -6q^2 \frac{(q^6; q^6)_\infty^2}{(q^{12}; q^{12})_\infty} a(q^3)^6 \psi(-q^9)(q^9; q^9)_\infty^3 \pmod{27}. \quad (3.6)$$

Thus

$$J(q) \equiv -6 \frac{(q^2; q^2)_\infty^2}{(q^4; q^4)_\infty} a(q)^6 \psi(-q^3)(q^3; q^3)_\infty^3 \pmod{27}.$$

From the product representations of  $X(q)$  and  $\psi(q)$ , it is easy to see that  $I(q) \equiv J(q) \pmod{27}$ .

In the same way, applying (2.2) and extracting the terms of the form  $q^{3n+2}$  in

$$-18qa(q)^5 \frac{(q^3; q^3)_\infty^{12} (q^{12}; q^{12})_\infty^2}{(q^3; q^3)_\infty^8 (q^6; q^6)_\infty} \equiv -18qa(q^3)^5 \frac{(q^3; q^3)_\infty^4 (q^{12}; q^{12})_\infty^2}{(q^6; q^6)_\infty} \pmod{27},$$

we get

$$q^2 K(q^3) \equiv 0 \pmod{27}.$$

Thus

$$K(q) \equiv 0 \pmod{27}.$$

If we extract the terms of the form  $q^{3n+2}$  in (3.2), divide by  $q^2$  and replace  $q^3$  by  $q$ , we deduce that

$$\sum_{n \geq 0} c\phi_6(9n+7)q^n \equiv 18(I(q) + J(q) + K(q)) \equiv 36J(q) \pmod{243}. \quad (3.7)$$

Note that

$$\begin{aligned} J(q) & \equiv -6\varphi(-q^2)a(q^3)^6 \psi(-q^3)(q^3; q^3)_\infty^3 \\ & \equiv -6\left(\varphi(-q^{18}) - 2q^2 X(-q^6)\right) a(q^3)^6 \psi(-q^3)(q^3; q^3)_\infty^3 \pmod{27}. \end{aligned} \quad (3.8)$$

Thus the terms of the form  $q^{3n+1}$  in  $J(q)$  vanish modulo 27. Therefore, by (3.7) we see that

$$c\phi_6(27n+16) \equiv 0 \pmod{243}.$$

Furthermore, by (3.8), extracting the terms of the form  $q^{3n}$  in (3.7), then replacing  $q^3$  by  $q$ , we obtain

$$\begin{aligned} & \sum_{n \geq 0} c\phi_6(27n+7)q^n \\ & \equiv -216\varphi(-q^6)a(q)^6\psi(-q)(q;q)_\infty^3 \\ & \equiv 27\varphi(-q^6)a(q^3)^6\left(Y(-q^3)-q\psi(-q^9)\right)\left(a(q^3)(q^3;q^3)_\infty-3q(q^9;q^9)_\infty^3\right) \pmod{243}. \end{aligned}$$

Extracting the terms of the form  $q^{3n+2}$ , dividing by  $q^2$  and replacing  $q^3$  by  $q$ , we get

$$\begin{aligned} & \sum_{n \geq 0} c\phi_6(81n+61)q^n \\ & \equiv 81\varphi(-q^2)a(q)^6\psi(-q^3)(q^3;q^3)_\infty^3 \\ & \equiv 81\left(\varphi(-q^{18})-2q^2X(-q^6)\right)a(q^3)^6\psi(-q^3)(q^3;q^3)_\infty^3 \pmod{243}. \end{aligned} \tag{3.9}$$

This implies (1.8). Note that the terms of the form  $q^{3n+1}$  do not appear in the right hand side of (3.9), we deduce that

$$c\phi_6(243n+142) \equiv 0 \pmod{243}.$$

From (3.7)–(3.9) we deduce that

$$c\phi_6(81n+61) \equiv 3c\phi_6(9n+7) \pmod{243}.$$

Replacing  $n$  by  $9n+6$  in this congruence relation and applying (1.8), we obtain

$$c\phi_6(729n+547) \equiv 0 \pmod{243}.$$

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